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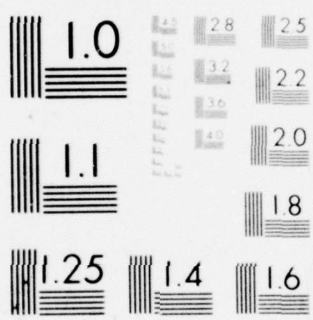


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# ASYMPTOTIC PROBABILITIES IN A CRITICAL AGE-DEPENDENT BRANCHING PROCESS

by

10 Howard J. Weiner

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Asymptotic Probabilities in a

Critical Age-Dependent Branching Process

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ABSTRACT

Let $Z(t)$ denote the number of cells alive at time $t$ in a critical Bellman-Harris age-dependent branching process, that is, where the mean number of offspring per parent is one. A comparison method is used to show for $k \geq 1$, and a high-order moment condition on $G(t)$, where $G(t)$ is the cell lifetime distribution, that $\lim_{t \to \infty} t^2 P[Z(t)=k] = a_k > 0$, where $\{a_k\}$ are constants.

The method is also applied to the total progeny in the critical process.

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## 1. Introduction.

Let $Z(t)$ denote the number of cells alive at time $t$ in a critical Bellman-Harris [2] age-dependent branching process. That is, at time $t = 0$, the process starts with one new cell with lifetime distribution $G(t)$, $G(0+) = 0$, non-lattice, and

$$G'(t) = g(t), \tag{1.1}$$

the density exists, and for some $\delta > 0$,

$$\int_0^\infty t^{4+\delta} g(t)dt < \infty. \tag{1.2}$$

At the end of its life, the cell disappears and is immediately replaced by $k$ new cells with probability $p_k \geq 0$, with

$$\sum_{k=0}^{\infty} p_k = 1 \tag{1.3}$$

and criticality,

$$\sum_{k=1}^{\infty} kp_k = 1. \tag{1.4}$$

Each new cell proceeds identically as the parent cell and independently of all other cells and the state of the system. Assume that for all integers $k$,

$$\sum_{n=1}^{\infty} n^k p_n < \infty. \tag{1.5}$$

It is the purpose of this note to show that for $k \geq 1$,

$$\lim_{t \to \infty} t^2 P[Z(t)=k] = a_k > 0, \tag{1.6}$$

where $\{a_k\}$ are constants.

The method is also applied to total progeny in the critical process.

## 2. Integral Equation and Result.

Define the offspring generating function, for $0 \leq s \leq 1$,

$$h(s) \equiv \sum_{k=0}^{\infty} p_k s^k. \tag{2.1}$$

Define the generating function for $Z(t)$ as, for $0 \leq s \leq 1$, all $t \geq 0$,

$$F(s,t) \equiv E(s^{Z(t)}). \tag{2.2}$$

Then [2] $F(s,t)$ satisfies

$$F(s,t) = 1 - G(t) + \int_0^t h(F(s,t-u))dG(u), \tag{2.3}$$

with $F(s,0) = s$.

Theorem. Let $Z(t)$ denote the number of cells at time $t$ in a critical Bellman-Harris branching process satisfying (1.1) - (1.5). Assume also that the derivatives of $h$ satisfy

$$0 < h''(1) < \infty, \quad 0 < h'''(1) < \infty. \tag{2.4}$$

Then for $k \geq 1$

$$\lim_{t \to \infty} t^2 P[Z(t)=k] = a_k > 0 \tag{2.5}$$

where $\{a_k\}$ are constants.

Proof. Consider first the case $k = 1$, and write for simplicity

(2.6) $$P(t) \equiv P[Z(t)=1].$$

From the representation (2.3), derivatives of $F(s,t)$ with respect to $s$ exist, and note that

(2.7) $$\left.\frac{\partial F(s,t)}{\partial s}\right|_{s=0} = P(t).$$

From (2.7), (2.3) one obtains the relationship

(2.8) $$P(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))P(t-u)dG(u)$$

and $P(0) = 1$,

where

(2.9) $$Q(t) = P[Z(t) > 0].$$

From [2],

(2.10) $$\lim_{t\to\infty} tQ(t) = 2\mu(h''(1))^{-1},$$

where

(2.11) $$\mu \equiv \int_0^\infty t dG(t).$$

Assume for simplicity in all the following that

(2.12) $$\mu \equiv 1.$$

The proof will proceed by a number of claims.

Claim I. Let $R(t)$ be continuous, $R(0) = 0$, and satisfy, for $t > 0$, that

$$(2.13) \qquad R(t) \underset{(<)}{>} \int_0^t h'(1-Q(t-u))R(t-u)dG(u).$$

Then for all $t > 0$,

$$(2.14) \qquad R(t) \underset{(<)}{>} 0.$$

Proof. Assume the upper inequality of (2.13).

Note first that for all $t > 0$,

$$(2.15) \qquad R(t) \neq 0.$$

This follows from (2.13) by assuming there is a $t_0$ such that $R(t_0) = 0$, and a contradiction is clear.

If the upper inequality in (2.13) is false, (2.15) requires that $R(t) < 0$ for $t > 0$. Then on an arbitrary interval $[0,t]$, assume that $R(t)$ assumes its minimum at $t_1 \leq t$. Since $0 \leq h'(x) \leq 1$, setting $t = t_1$ in the upper inequality (2.13) yields a contradiction. The lower inequality of (2.13) implies the lower inequality of (2.14) similarly, completing the claim.

Claim II. For some $0 < \epsilon < 1$, all $t \geq 0$, define

$$(2.16) \qquad K(t) = \begin{cases} 1 & , \quad t < 1 \\ \dfrac{1}{t^{2-\epsilon}} & , \quad t \geq 1 \end{cases}$$

$$(2.17) \qquad L(t) = \begin{cases} 1 & , \quad t < 1 \\ \dfrac{1}{t^{2+\epsilon}} & , \quad t \geq 1. \end{cases}$$

Then for all $t$ sufficiently large,

$$(2.18)\qquad K(t) > 1 - G(t) + \int_0^t h'(1-Q(t-u))K(t-u)dG(u)$$

and

$$(2.19)\qquad L(t) < 1 - G(t) + \int_0^t h'(1-Q(t-u))L(t-u)dG(u).$$

<u>Proof</u>. To show (2.18), note that the right side yields the inequality

$$(2.20)\qquad \int_0^t h'(1-Q(t-u))K(t-u)dG(u) \leq G(t) - G(t/2)$$

$$+ \int_0^{t/2} h'(1-Q(t-u))K(t-u)dG(u).$$

A Taylor expansion of the rhs of (2.20) yields,

$$(2.21)\qquad \int_0^{t/2} h'(1-Q(t-u))K(t-u)dG(u)$$

$$= \int_0^{t/2} (1-Q(t-u)h''(1) + o(Q(t-u)))K(t-u)dG(u).$$

A Newton binomial expansion applied to the rhs integrand of (2.21) using (2.10), Theorem 4, p. 406 of [3], (2.16), and that $\mu \equiv 1$ yields that the rhs of (2.21) equals

$$(2.22) \qquad \int_0^{t/2} \left(1 - \frac{2}{t-u} + o\left(\frac{1}{t-u}\right)\right) \frac{dG(u)}{(t-u)^{2-\epsilon}}$$

$$= \frac{1}{t^{2-\epsilon}} \int_0^{t/2} \frac{dG(u)}{(1 - \frac{u}{t})^{2-\epsilon}} - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right)$$

$$= \frac{1}{t^{2-\epsilon}} \int_0^{t/2} \left(1 + \frac{(2-\epsilon)u}{t} + o\left(\frac{1}{t}\right)\right) dG(u) - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right)$$

$$= \frac{1}{t^{2-\epsilon}} + \frac{(2-\epsilon)}{t^{3-\epsilon}} - \frac{2}{t^{3-\epsilon}} + o\left(\frac{1}{t^{3-\epsilon}}\right)$$

$$< \frac{1}{t^{2-\epsilon}}$$

for all $t$ sufficiently large. In view of (2.20), (1.1), this suffices for (2.18), and (2.19) follows similarly.

Claim III. For $0 < \epsilon < 1$, and all $t$ sufficiently large,

$$(2.23) \qquad \frac{1}{t^{2+\epsilon}} < P(t) < \frac{1}{t^{2-\epsilon}} .$$

Proof. This follows from Claims I and II, where $R = K - P$, for example.

Claim IV. Let $R(t)$ satisfy

$$(2.24) \qquad R(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))R(t-u)dG_o(u),$$

with $R(0) = 1$, where $Q(t)$ is the probability of non-extinction for the process with lifetime $G(t)$, and

$$(2.25) \qquad 1 - G_o(t) = e^{-t}.$$

Then as $t \to \infty$,

(2.26) $$R(t) \sim \frac{c}{t^2}$$

for some constant $c > 0$.

Proof. Write (2.24) as

(2.27) $$R(t) = 1 - G(t) + e^{-t}\int_0^t h'(1-Q(u))R(u)e^u du.$$

Differentiating, with respect to $t$,

(2.28) $$R'(t) = -g(t) + (1-G(t)-R(t)) + h'(1-Q(t))R(t).$$

Then, expanding $h$ in a Taylor series, one obtains

(2.29) $$R'(t) + \frac{2}{t}R(t) = f(t)$$

where

(2.30) $$f(t) = 1 - G(t) - g(t) + \frac{h'''(*)}{2}Q^2(t)R(t) + (Q(t) - \frac{2}{h''(1)t})R(t),$$

with $1 - Q(t) \leq * \leq 1$. In view of (1.1), (1.2), (1.5), it follows from Theorem 4 of [3], p. 406, that

(2.31) $$(Q(t) - \frac{2}{h''(1)t}) = O(\frac{\log t}{t^2}).$$

An analysis similar to that in Claims I - III yields that for $0 < \epsilon < 1$, for $t$ sufficiently large,

$$\frac{1}{t^{2+\epsilon}} < R(t) < \frac{1}{t^{2-\epsilon}} . \tag{2.32}$$

Then (1.1), (2.31), (2.32) used in (2.30) imply that

$$t^2 f(t) \tag{2.33}$$

is integrable with respect to Lebesgue measure on the positive line.

To solve (2.29) for $t$ large, let

$$Q(t) \equiv t^2 R(t). \tag{2.34}$$

Then (2.29) becomes

$$Q'(t) = t^2 f(t). \tag{2.35}$$

For fixed $t_o > 0$, one then obtains

$$Q(t) - Q(t_o) = \int_{t_o}^{t} \xi^2 f(\xi) d\xi \tag{2.36}$$

or

$$R(t) \sim \frac{c}{t^2} \tag{2.37}$$

for $t$ sufficiently large, and $c > 0$, completing this claim.

Claim V. Let

$$(2.38)\quad T(t) = 1 - G(t) + \int_0^t h'(1-Q(t-u))R(t-u)dG(u).$$

Then, as $t \to \infty$,

$$(2.39)\qquad t^2|T(t)-R(t)| \to 0.$$

Proof.

$$(2.40)\quad |T(t)-R(t)| \le \left|\int_0^{t/2} h'(1-Q(t-u))R(t-u)(dG(u)-dG_o(u)\right|$$

$$+ G(t) - G(t/2) + G_o(t) - G_o(t/2).$$

A Taylor series and Newton binomial expansion on the right side integrand of (2.40) yields

$$(2.41)\quad \int_0^{t/2}\left[1-Q(t-u)h''(1) + Q^2(t-u)h'''(k(t-u))\right]\left[\frac{c}{(t-u)^2} + o(\frac{1}{(t-u)^2})\right]\left(dG(u)-dG_o(u)\right)$$

where

$$(2.42)\qquad 1 - Q(t-u) \le k(t-u) \le 1,$$

and expression (2.41) equals

$$(2.43)\quad \int_0^{t/2}\left[1-\frac{2}{t-u}+o\left(\frac{1}{t-u}\right)\right]\left[\frac{c}{(t-u)^2}+o\left(\frac{1}{(t-u)^2}\right)\right]\left(dG(u)-dG_o(u)\right)$$

$$=\int_0^{t/2}\left[1-\frac{2}{t}\left(1+\frac{u}{t}\right)+o\left(\frac{1}{t}\right)\right]\left[\frac{c}{t^2}+o\left(\frac{1}{t}\right)\right]\left(dG(u)-dG_o(u)\right)$$

$$=o(t^{-4}),$$

for $t$ sufficiently large, using that

$$\mu=\int_0^\infty u\,dG(u)=\int_0^\infty u\,dG_o(u)=1.$$

This completes Claim V.

The proof of the theorem for P(t) may now be completed. Define the iterative scheme, for $n \geq 0$

$$(2.44)\quad P_{(n+1)}(t)=1-G(t)+\int_0^t h'(1-Q(t-u))P_{(n)}(t-u)\,dG(u),$$

and let

$$(2.45)\quad P_{(0)}(t)\equiv R(t).$$

Then from (2.38), it follows that

$$(2.46)\quad P_{(1)}(t)=T(t).$$

Claims IV and V imply that, for $t$ large,

$$(2.47)\quad P_{(0)}(t)\sim\frac{c}{t^2}\ \text{ and }\ P_{(1)}(t)\sim\frac{c}{t^2}\,.$$

An induction argument along the lines of Claim V establishes that, for large $t$,

$$P_{(n)}(t) \sim \frac{c}{t^2} . \tag{2.48}$$

Note that for $n \geq 0$,

$$P(t) - P_{(n+1)}(t) = \int_0^t h'(1-Q(t-u))(P(t-u)-P_{(n)}(t-u))dG(u). \tag{2.49}$$

Denote

$$\Delta_n(t) \equiv P(t) - P_{(n)}(t), \tag{2.50}$$

then from (2.49),

$$|\Delta_{n+1}(t)| < \int_0^t |\Delta_n(t-u)|dG(u) < |\Delta_o|^* G_{n+1}(t), \tag{2.51}$$

where $*$ denotes convolution, and $G_n(t)$ is the $n^{th}$ convolution of $G$ with itself.

Let $\{X_i\}$ denote I.I.D. random variables, with distribution function $G(t)$. Set

$$S_n \equiv \sum_{i=1}^{n} X_i. \tag{2.52}$$

Then

$$G_n(t) \equiv P[S_n \leq t] = P[S_n - n \leq t-n], \tag{2.53}$$

and if one takes $n > t$, by (1.2) and Chebyshev's inequality,

$$G_n(t) \le \frac{\mathrm{Var}\, S_n}{(n-t)^2} = \frac{n}{(n-t)^2} \, . \tag{2.54}$$

Also,

$$|\Delta_o| * G_n(t) \le G(t) - G(t/2) + \int_0^{t/2} |P(t-u)-R(t-u)| dG_n(u) \tag{2.55}$$

and (1.2), Claims III, Iv, and (2.54) yields that

$$|\Delta_o| * G_n(t) \le \frac{Kn}{t^{2-\epsilon}(n-t)^2} \, , \tag{2.56}$$

where $K$ is a constant.

For fixed large $t$, let

$$n > [t^{1+\epsilon}]. \tag{2.57}$$

This suffices for the proof of the theorem for $P(t) \equiv P[Z(t) = 1]$.

For $k \ge 2$, note that

$$k!P[Z(t)=k] \equiv k!P_k(t) = \left.\frac{\partial^k F(s,t)}{\partial s}\right|_{s=0} \, . \tag{2.58}$$

Then (2.58) applied to (2.3) for $k = 2$ yields

$$P_2(t) = \frac{1}{2}\int_0^t h''(1-Q(t-u))P^2(t-u)dG(u) + \int_0^t h'(1-Q(t-u))P_2(t-u)dG(u). \tag{2.59}$$

Since the theorem is true for $P(t)$, then as (2.59) is of the same form as (2.8), a similar analysis as that used to establish the result for $P(t)$ also establishes it for $P_2(t)$. Assume the result true for $P_k(t)$, $k \leq n$. Applying (2.58) to (2.3) for $k = n+1$ yields an equation of the form

$$(2.60) \qquad P_{n+1}(t) = f_{n+1}(t) + \int_0^t h'(1-Q(t-u))P_{n+1}(t-u)dG(u)$$

where the induction hypothesis yields that

$$(2.61) \qquad f(t) = o(t^{-4}).$$

In view of (2.61), equation (2.60) is of the same form as (2.8), and the analysis for $P(t)$ applies to $P_{n+1}(t)$, yielding the theorem.

Remark. This result is known for the critical discrete-time or Galton-Watson process. See [2] Ch. 1, for example. It may be possible that a series of comparison sequences of the type given in (2.44), (2.45) relating the critical Bellman-Harris process with a corresponding critical Galton-Watson process could yield this result.

## 3. Total Progeny.

For another application of the method, let

(3.1) $N(t)$ = total number of progeny born by $t$ in a critical age-dependent branching process with $\infty > h'(1) \equiv \Sigma_{k=1}^{\infty} kp_k > 0$.

Denote, for $0 \leq s \leq 1$, $t \geq 0$

$$(3.2) \qquad H(s,t) \equiv E(s^{N(t)}) = \Sigma_{k=1}^{\infty} P[N(t)=k]s^k.$$

Then (e.g. [4], p. 394)

$$(3.3) \qquad H(s,t) = s\left[1 - G(t) + \int_0^t h(H(s,t-u))dG(u)\right].$$

Since, for $k \geq 1$,

$$(3.4) \qquad \left.\frac{\partial^k H(s,t)}{\partial s^k}\right|_{s=0} = k!P[N(t)=k] \equiv k!Q_k(t)$$

one may apply (3.4) to (3. 3) to obtain the limiting values of $P[N(t)=k] \equiv Q_k(t)$.

For $k = 1, 2$ one obtains, since $H(0,t) \equiv 0$, that

$$(3.5) \qquad Q_1(t) = 1 - G(t) + \int_0^t h(0)dG(u) \to p_0 \quad \text{as } t \to \infty.$$

$$(3.6) \qquad Q_2(t) = \int_0^{t/2} + \int_{t/2}^t h'(0)Q_1(t-u)dG(u) \to p_0p_1 \quad \text{as } t \to \infty.$$

Similarly, by induction and an application of Leibniz' lemma for successive differentiation, the limiting values of the $\{Q_k(t)\}$ may be obtained. This result is implied by a representation essentially as that in [1], pp. 275-276. No assumptions on higher moments of $G$ are required, but all moments of $h(s)$ are needed in this approach.

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* in the limit as $t$ approaches infinity $t$ squared P[...